Large deviation function for the Eden model and universality within the one-dimensional Kardar-Parisi-Zhang class

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It has been recently conjectured that for large systems, the shape of the central part of the large deviation function of the growth velocity would be universal for all the growth systems described by the Kardar-Parisi-Zhang equation in 1+1 dimension. One signature of this universality would be that the ratio of cumulants $\mathcal{R}_t=[\langle h_t^3\rangle_c]^2/[\langle h_t^2\rangle_c\langle h_t^4\rangle_c]$ would tend towards a universal value 0.415 17 . . . as t tends to infinity, provided periodic boundary conditions are used. This has recently been questioned by Stauffer. In this paper we summarize various numerical and analytical results supporting this conjecture, and report in particular some numerical measurements of the ratio $\mathcal{R}_t$ for the Eden model.

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I. INTRODUCTION

Phenomena ranging from growth processes to directed polymers in random media can be described on a large scale by the Kardar-Parisi-Zhang (KPZ) equation. In previous papers Derrida *et al.* [1,2] have conjectured that for all (1+1)-dimensional models within the KPZ class, the large deviation function associated with the height h_t of the interface follows some universal scaling in the limit of large systems.

If this conjecture is true, then it implies in particular that for periodic boundary conditions, the ratio of cumulants

$$\mathcal{R}_t=\frac{[\langle h_t^3\rangle_c]^2}{\langle h_t^2\rangle_c\langle h_t^4\rangle_c} \tag{1}$$

converges towards a universal value 0.415 17 . . . as t tends to infinity. Here, h_t is the space-averaged height, and $\langle \cdots \rangle$ refers to ensemble averages.

In answer to some objections this conjecture raised, we find it useful to explain more precisely in which frame we expect universality. It will also be an opportunity to summarize how the conjecture has been verified in some other models, since our last publication [3].

In Sec. IV, we shall explain why the numerical results presented by Stauffer [4] for the Eden model are in fact not in contradiction with our conjecture. In order to sustain our claim, we present numerical results for the Eden model that have been performed in the proper frame.

II. CONJECTURE OF UNIVERSALITY

In this section, we shall first recall the content of the conjecture. Though the KPZ equation can describe systems as different as particles moving on a lattice or directed polymers in random media, we shall consider only growth models, in order to simplify the presentation. The extension to other applications should be straightforward.

We consider a discrete growth model on a one-dimensional lattice of N sites, with periodic boundary conditions. At each time step, a growth event—whatever it is—occurs in each site with probability dt.

The quantity h_t whose distribution we are interested in is the space-averaged height after t time steps. Note that in [1,2], the variable under consideration was rather $Y_t=Nh_t$. That is why the formulation of the results may slightly differ here. Equivalently, we shall use as a variable, the averaged velocity $v_t=h_t/t$.

If t is large enough, the probability distribution $P(h_t)$ should become independent of the initial condition. In the long time limit, the large deviation function f is defined as

$$f(v)=\lim_{t\to\infty}\frac{\ln P(vt)}{t}. \tag{2}$$

For large systems, we conjecture that when the deviation of v from its average $\overline{v}$ is at most of order $1/N$, the large deviation function takes the form

$$f(v)=\mathcal{K}H\left(N\frac{v-\overline{v}}{\overline{v}}\right), \tag{3}$$

where H has the following asymptotic behavior:

$$H(V)=-V^2+O(V)^3 \quad \text{for} \quad |V|\ll 1,$$

$$H(V)\simeq-[2\sqrt{3}/(5\sqrt{\pi})]V^{5/2} \quad \text{for} \quad V\to+\infty,$$

$$H(V)\simeq-[4\sqrt{\pi}/3]|V|^{3/2} \quad \text{for} \quad V\to-\infty.$$

The coefficient $\mathcal{K}$ is defined as

$$\mathcal{K}=\frac{1}{2N^2}\frac{\overline{v^2}}{\lim_{t\to\infty}(\langle h_t^2\rangle_c/t)}, \tag{4}$$

where $\langle h_t^2\rangle_c$ is the second cumulant. The rescaling factor $\mathcal{K}$ is model dependant, but the shape H is expected to be the same for all microscopic models belonging to the KPZ class.

Note that the shape H given above may reflect the type of boundary conditions we use. For example, we may have a

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different result for open boundary conditions. However, H is conjectured to be universal in the sense that it would be the same for *any* microscopic model, given that we keep the same geometrical constraints—in our case periodic boundary conditions.

The knowledge of the central part of the large deviation function determines all the cumulants $\langle h_t^n \rangle_c$, as explained in the Appendix. The universality of the limit of Eq. (1) is a direct consequence of the universality of the shape Eq. (3).

III. ANALYTICAL VERIFICATION OF THE CONJECTURE

In this section, we summarize the analytical results that have been obtained for the large deviation function on different models. First, the conjecture had been proposed after the large deviation function had been calculated for one special model of the KPZ class, namely, the asymmetric exclusion process (ASEP).

In the ASEP, one considers a system of p particles moving on a ring of N sites. During every time interval dt, each particle jumps to the next site on its right with probability dt, if this site is empty. Otherwise, it does not move.

Of course, for this model, the large deviation function [2] does verify the scaling (3), with

$$\mathcal{K}=\sqrt{\frac{\rho(1-\rho)}{\pi N^3}}$$

and

$$\overline{v}=\frac{N}{N-1}\rho(1-\rho).$$

Since then, Lee and Kim [5] have extended the result to the partially asymmetric exclusion process, i.e., to the case where particles are allowed to jump either to the right or to the left with probabilities $(1+\epsilon)dt/2$ or $(1-\epsilon)dt/2$. To do so, they have used the formalism of quantum spin chains, and found again the form (3) with

$$\mathcal{K}=\epsilon\sqrt{\frac{\rho(1-\rho)}{\pi N^3}}$$

and

$$\overline{v}=\epsilon\frac{N}{N-1}\rho(1-\rho).$$

Brunet and Derrida have been considering directed polymers pinned on impurities. This model belongs to the KPZ class if we consider that the height of the interface corresponds now to the free energy for a polymer of length t. According to some preliminary results, it seems that the scaling Eq. (3) for the large deviation function, and thus also the universal asymptotic value for $\mathcal{R}_t$, are verified [6].

IV. NUMERICAL MEASUREMENTS FOR THE RATIO OF CUMULANTS $\mathcal{R}_T$

For most models, however, analytical results are hard to obtain. It is more convenient to check numerically whether the ratio of cumulant $\mathcal{R}_t$ converges towards the conjectured universal value 0.415 17 . . . when t tends to infinity, in the limit of a large system.

It must be noticed that, as we are interested in the statistics up to the fourth moment, of a quantity that is itself an average over a huge number of data, this calculation is enormously demanding in terms of computer time. Besides, to determine $\mathcal{R}_t$ from a given simulation, we have less and less statistics as t increases. This is why fluctuations become important for large t.

Thus, our numerical calculations are more indications in favor of the conjecture, rather than real numerical proofs. However, we find them quite significant. In [2], some simulations had been performed for various deposition models.

Following this work, Stauffer had published [4] some numerical results obtained for the Eden model, which seemed in contradiction with the conjecture. We think there is in fact no contradiction. The discrepancy comes from the definition of time. If time is defined in the way we advocate below, then our numerical results are clearly compatible with the conjecture.

First we recall the definition of the Eden model. We still consider an interface growing on a ring with N sites. At each time step, we choose one of the boundary sites, i.e., an empty site that has at least one side in common with an occupied site (this would correspond to version A in Ref. [7]). When a boundary site is chosen, it becomes occupied and its empty neighbors become boundary sites.

As the shape of the interface varies with time, so does the number N_B of boundary sites. In the continuous limit, at each time step dt, each boundary site should be chosen with probability dt. This means that the larger N_B, the more probable it is that at least one site will be chosen.

In order to take this effect into account in numerical simulations, time should not be incremented by a constant amount between two choices of a boundary site. The time increment should be weighted by $1/N_B$ [8].

This is the main difference between our numerical simulations and Stauffer's. Stauffer was using the original version of the Eden model [9], where time is not weighted. With the weighted definition of time, we find that our simulations presented in Fig. 1 are compatible with the conjecture.

Stauffer's results give an indication that the original Eden model would not belong to the KPZ class. This could be expected from the following simple argument. As we explained above, taking a weighted time implies that an event may occur in each site independently. If we take a nonweighted time, this implies implicitly that the probability for an event to occur in a given site depends on the shape of the interface in the whole system. Thus the growth rules are not local any more, and it is not so surprising that the model is not in the KPZ class.

V. CONCLUSION

After rephrasing the universality conjecture of [1], we have summarized several analytical and numerical results obtained for different models that confront the conjecture. We have presented here some simulation results indicating that also in the case of the Eden model, the cumulant ratio $\mathcal{R}_t$ tends towards the conjectured value 0.415 17 . . . in the large

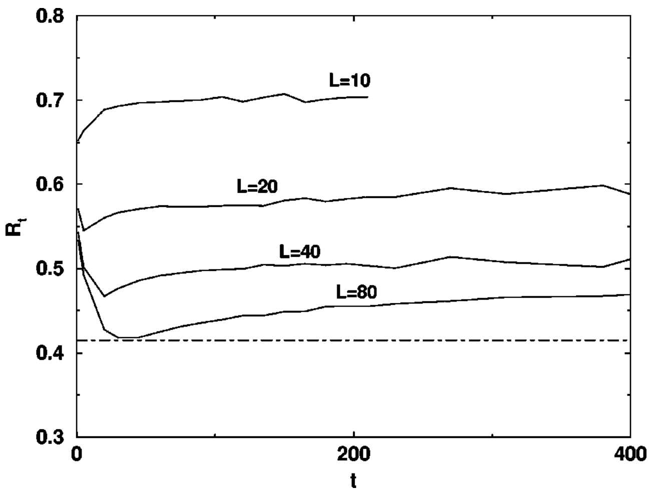

FIG. 1. Cumulant ratio $\mathcal{R}_t$ as a function of time for different system sizes. The horizontal dot-dashed line gives the expected asymptotic value for an infinite system in the infinite time limit. The statistics were accumulated, respectively, during 10^{11}, 8.5×10^{10}, 2.3×10^{10}, and 8×10^{9} time units, for the system sizes $L=10$, $L=20$, $L=40$, and $L=80$. Time is defined in such a way that at each time step dt, each boundary site may be chosen with probability dt.

time limit and for large systems, provided that the proper definition of time is chosen.

Here we have considered only sequential dynamics. *A priori*, we would expect the same scaling if parallel dynamics were used, as long as the process remains stochastic (if heights would grow in parallel with probability one, of course no fluctuations would appear with time). It would be interesting to have numerical or analytical results in this case.

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APPENDIX

As indicated in Ref. [2], the cumulants in the long time limit are obtained as

$$\lim_{t\to\infty}\frac{\langle h_t^n\rangle_c}{t}=\left.\frac{d^n\lambda(\alpha)}{d\alpha^n}\right|_{\alpha=0}, \tag{A1}$$

where the function $\lambda(\alpha)$ is related to the large deviation function $f(v)$ by

$$\lambda(\alpha)=\max_v[f(v)+\alpha v]. \tag{A2}$$

For $\alpha=0$, the maximum is achieved for the mean velocity $\tilde{v}=\overline{v}$. For general α, the velocity $\tilde{v}$ corresponding to the maximum is found as a solution of

$$\alpha+f'(\tilde{v})=0. \tag{A3}$$

For small α, this equation can be solved by expanding $\tilde{v}-\overline{v}$ in powers of α. Reporting the result into Eq. (A2) yields an expansion of $\lambda(\alpha)$ in powers of α whose coefficients are—up to simple factors—the cumulants. In particular, the first two terms of the expansion just give the definition of the mean velocity $\langle h_t\rangle=\overline{v}$ and the definition (4) of K.

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